Asymptotic Expansions with Oscillating Coefficients

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1 Introduction

In recent years, Computer Algebra has seen significant advances on a wide range of fronts. One of the many areas of development has been Symbolic Asymptotics.
The *exp–log functions* are those defined by expressions built from the rational numbers $\mathbb{Q}$ and the variable, x, using arithmetic operations and the functions exp and log, with the understanding that the latter is only applied to arguments which are eventually positive. Modulo difficulties with signs of constants, algorithms exist to determine the asymptotics of exp–log functions, [11, 31, 21]. Moreover one can add integration and extraction of algebraic roots to the signature, [37], and likewise composition with functions which are given by ordinary differential equations and which are meromorphic at the limit, [35]. Inverse functions can be handled, [28], and expansions of implicit functions can be obtained, [27, 39]. In addition there is substantial progress with Hardy–field solutions of differential equations, [33, 36, 39]. Practical development has been slower to follow, but there are now implementations of the exp–log algorithm in Maple, by Dominik Gruntz [14], and in Aldor by James Beaumont. Moreover the multiseries algorithm for inverse functions has been implemented in Maple [28] and used to give new results in combinatorics.

To date the omission of trigonometric and other oscillating functions from the theory represents a major gap, and one which is particularly to be regretted since asymptotic expansions associated with the differential equations of mathematical physics very typically involve sines and cosines. Their absence has not been mere oversight.
Firstly the major part of the existing development is based on the theory of Hardy fields (see Appendix 1), but a non–zero element of a Hardy field cannot have arbitrary large zeros. Moreover if division is to be included in the signature, sines and cosines will give rise to infinitely many singularities. In [1] it is shown that for ϕ an arbitrary monotone increasing function, there exists a choice of $\alpha \in \mathbb{R}$ such that the function $u(t) = (2 - \cos t - \cos(\alpha t))^{-1}$ is $\mathcal{C}^\infty$ and has the property that $\limsup\{\mathrm{u(t)}/\phi(\mathrm{t})\} \geq 1$.
Secondly if unrestricted composition of sines is allowed in expressions, it is known that there is no algorithm to decide whether the function defined by the given expression tends to a limit, [19].
Of course it has long been known that the theory of real functions presents many pathologies. A triumph of the Lesbesgue theory has been the demonstration that things become much easier, particularly in the domain of integration, if one is prepared to ignore bad behaviour on a relatively small set. Naturally there is a price to pay. When two functions differing

only on a set of measure zero are deemed equivalent, one loses the notion of the value of a function at a point, which is perhaps the most obvious characteristic of a function!
A major purpose of the present paper is to show that something similar holds in algorithmic theory of limits. If one is prepared to ignore what happens on a relatively small set one can obtain asymptotic expansions similar to multiseries (see below) while allowing trigonometric functions into the signature in a non–trivial way. Part of the price to be paid is that coefficients in expansions need no longer be constant, nor even tend to a limit. However one can assert that they are bounded and bounded away from zero off the bad set. Of course the expansion does not tell us what happens on the bad set.

In Section 2, we define the set of coefficient functions and show that this set contains the field of functions $\mathbb{R}(\sin b_1(x), \sin b_2(x), \ldots, \sin b_k(x))$; here $k \in \mathbb{N}$ and $b_1, \ldots, b_k$ are elements of a Hardy field which tend to infinity with x (subject to one natural restriction). In Section 3 we give a brief outline of the existing theory of multiseries, and then in the following section we consider the functions defined by expressions with signature $\mathbb{R}, x, +, -, \times, \div, \exp, \log, \sin$ with the proviso that sines are not permitted to appear inside the arguments of the transcendental functions. We give an algorithm to obtain a multiseries–type expansion of such a function with coefficients given in closed form and lying in the designated set of coefficient functions. This section concludes with two examples.
Finally in the Appendix, we give a very brief introduction to Hardy fields.

Our main aim in this paper has been to introduce ideas, and so we have not striven for the most general and powerful theory. Thus it would almost certainly be possible to bring integration into the signature of the function class considered in Section 4. Similarly there might be more powerful definitions embodying the idea of a wandering function, given in Section 2.

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2 Coefficient Classes

The idea of a *wandering function* is that the values of the function do not especially favour the neighbourhoods of any particular point including the 'point at infinity'.
Let μ be Lesbesgue measure on $\mathbb{R}$. We write $\mathcal{W}$ for the set of functions, f, defined on (a, ∞) for some $a \in \mathbb{R}$, except perhaps on a set P_f, such that for any $v \in \mathbb{R}$

$$\lim_{b\to\infty} \lim_{\delta\to 0} \mathrm{limsup}_{T\to\infty} \frac{\mu((\{\delta < |f - v| < \delta^{-1}\} \cap [b, T]) \setminus P_f))}{\mu([b, T])} = 1. \tag{1}$$

$\mathcal{W}$ is our candidate for the set of *wandering functions*. The definition expresses the idea that $f - v$ is mostly bounded and bounded away from zero, and has some parallels with the notion of convergence in measure, [15]. Key considerations in its framing were the need to include the non–constant coefficient functions appearing in expansions of elementary functions and the wish for a reasonable level of generality. The first criterion requires us to take the limsup over T rather than just the limit, as a later example will make clear (see the end of Section 2). Of course $\mathcal{W} \cap \mathbb{R} = \emptyset$, but if $f \in \mathcal{W}$ and c is a real constant then $f + c \in \mathcal{W}$.

A desirable property of the definition would be that $\mathcal{W}$ be closed under composition with Hardy–field elements tending to infinity. Unfortunately the present definition does not achieve this. For example if

$$f(x) = 1 + (-1)^{[\log\log x]}$$

and we take $T_n = \exp(e^{2n+1}) - 1$ and $t_n = \exp(e^{2n})$ then with μ being Lesbegue measure,

$$\frac{\mu(\{1/3 < |f| < 3\} \cap [e, T_n])}{\mu([e, T_n])} \geq \frac{T_n - t_n}{T_n}. \tag{2}$$

Now

$$T_n/t_n = \frac{\exp(e^{2n+1} - 1)}{\exp(e^{2n})} = \exp((e-1)e^{2n} - 1) \to \infty,$$

and it follows that the right–hand side of (2) tends to 1. So $f \in \mathcal{W}$. However if we take $g = \exp(\exp x)$ then $f \circ g(x) = 1 + (-1)^{[x]}$ and clearly

$$\frac{\mu(\{f \circ g = 0\} \cap [0, T])}{\mu([0, T])} \to \frac{1}{2}.$$

It may be that further research will yield a better definition of wandering functions, which remains natural and gives a class which is closed under scaling.

2.1 Combinations of Trigonometric Functions

Let $b_1(x)$,…,$b_k(x)$ be elements of a Hardy field which tend to infinity such that $b_i/b_{i-1} \to 0$ for $2 \leq i \leq k$. Let $J_i \in \mathbb{N}$ and let $\lambda_{i,j}, \nu_{i,j} \in \mathbb{R}$ for $j = 1, \ldots, J_i$. We write

$$\mathcal{R} = \mathbb{R}(\sin(\lambda_{1,1}b_1(x)), \cos(\nu_{1,1}b_1(x)), \ldots, \cos(\nu_{1,J_1}b_1(x)), \ldots, \sin(\lambda_{k,J_k}b_k(x)), \cos(\nu_{k,J_k}b_k(x))).$$

$\mathcal{R}$ is our the field of coefficient functions in the expansions to be introduced in Section 3. The main aim now is to show that $\mathcal{R} \subset \mathcal{W} \cup \mathbb{R}$. In order to do this we introduce an intermediate set of functions, $\mathcal{S}$. This is the set of $\mathcal{C}^\infty$ functions, f, defined on some interval $(\alpha, +\infty) \subset \mathbb{R}$ except perhaps at a countable number of points such that either $f = 0$ or the following holds.

> For any $\varepsilon \in \mathbb{R}^+$ and any $l \in \mathbb{R}^+$ there exists $a = a(\varepsilon) \in \mathbb{R}$ and $m = m(\varepsilon, l), M = M(\varepsilon, l) \in \mathbb{R}^+$ such that in any finite interval $I \subset (a, +\infty)$ with $|I| \geq l$ there are sub-intervals S_1,…,S_N with $\sum_{j=1}^{N} |S_j| < \varepsilon |I|$ and
>
> $$m < |f(x)| < M \quad \forall x \in I \setminus \cup_{j=1}^{N} S_j.$$

The dependence of m and M on l is necessary to cater for small intervals containing a zero of f. We refer to the sets S_1,…,S_N as the *exceptional sets.* If there is an $m = m(\varepsilon, l)$ such that $|f| > m$ except on exceptional sets of relative total length less than ε, we say that f is *mainly bounded away from zero.* Similarly if $M(\varepsilon, l)$ exists such that $|f| < M$ except on exceptional sets, we say that f is *mainly bounded.* We shall see that if $f - v \in \mathcal{S}$ for all $v \in \mathbb{R}$ then $f \in \mathcal{W} \cup \mathbb{R}$.
In fact we do not have $\mathcal{R} \subset \mathcal{S}$ in general, but we can prove that this is the case when $\gamma_0(b_1(x)) \leq \gamma_0(x)$. A scaling argument then gives that $\mathcal{R} \subset \mathcal{W} \cup \mathbb{R}$ as required.

First we establish a number of lemmas.

Lemma 1 *Let $f, g \in \mathcal{C}^\infty(\mathbb{R}^+)$. If $f \in \mathcal{S}$ and $g \to 0$ as $x \to \infty$ then $f + g \in \mathcal{S}$.*

Proof of Lemma 1
Given $\varepsilon, l \in \mathbb{R}^+$, let a, m and M be the quantities that $\mathcal{S}$ gives for f. Choose $b \geq a$ in $\mathbb{R}$ such that $|g(x)| < m/2$ for $x \geq b$. For any $I \subset (b, \infty)$ with $|I| \geq l$, let $S_1, \ldots, S_k$ be the exceptional sets for f. Then $m/2 < |f + g| < M + m/2$ on $I \setminus \cup_{j=1}^N S_j$. This completes the proof.

Lemma 2 *If $f, g \in \mathcal{S}$ then $fg \in \mathcal{S}$.*

For the proof it suffices to observe that $m_1 m_2 < |fg| < M_1 M_2$ if $m_1 < |f| < M_1$ and $m_2 < |g| < M_2$.

Lemma 3 *A product of sines, $\prod_{j=1}^J \sin v_j$ may be written in the form*

$$\prod_{j=1}^{J} \sin v_j = \sum_{k=1}^{K} a_k \sin w_k,$$

where $a_1, \ldots, a_k \in \mathbb{R}$ and each w_k is of the form

$$w_k = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_j v_j + \beta,$$

with $\alpha_1, \ldots, \alpha_J, \beta \in \mathbb{R}$.

Proof of Lemma 3
This is standard and uses induction on J. The case $J = 1$ is trivial, so suppose the lemma holds for a given value of J. Then

$$\begin{aligned}\prod_{j=1}^{J+1} \sin v_j &= \left(\sum_{k=1}^{K} a_k \sin w_k\right) \sin v_{J+1} = \sum_{k=1}^{K} a_k \sin w_k \sin v_{J+1} \\ &= \sum_{k=1}^{K} \frac{a_k}{2} \left\{\sin\left(\frac{\pi}{2} + v_{J+1} - w_k\right) - \sin\left(\frac{\pi}{2} - v_{J+1} - w_k\right)\right\}.\end{aligned}$$

The lemma now follows.

Lemma 4 *Suppose that $|f''| \leq M$ on (a, ∞), where $M > 0$. Suppose that $|f'(x_0)| > m_1$, where $m_1 \in \mathbb{R}^+$, and suppose also that $x_0 - m_1/(2M) > a$. Then $|f| > m_1^3$ on $(x_0 - m_1/(2M), x_0 + m_1/(2M))$ except perhaps on a sub-interval of length at most $8m_1^2$.*

Proof of Lemma 4
If $|x - x_0| < m_1/(2M)$, the First Mean Value Theorem and the Triangle Inequality give

$$\begin{aligned}|f'(x)| &= |f'(x_0) + (x - x_0) f''(\zeta)| \\ &\geq |f'(x_0)| - |x - x_0||f''(\zeta)| > m_1 - M|x - x_0| > m_1/2,\end{aligned} \tag{3}$$

where ζ lies between x and x_0.
Next suppose that $|x_1 - x_0| < m_1/(2M)$ and that $|f(x_1)| < m_1^3$. Then for $|x - x_0| < m_1/(2M)$

$$|f(x)| \geq |x - x_1||f'(\zeta)| - |f(x_1)| > |x - x_1|\frac{m_1}{2} - m_1^3 > m_1^3 \tag{4}$$

if $|x - x_1| > 4m_1^2$. Thus $|f| > m_1^3$ on $(x_0 - m_1/(2M), x_0 + m_1/(2M))$ except perhaps on a single sub-interval of length at most $8m_1^2$, which proves Lemma 4.

Lemma **5** *Let $\alpha \in \mathbb{R}$ and let $f \in \mathcal{C}^2(\alpha, \infty)$. Suppose that f' is mainly bounded away from zero on (α, ∞) and f'' is bounded there. Then f is mainly bounded away from zero on (α, ∞). It follows that if such an f is bounded, it must belong to $\mathcal{S}$.*

Proof of Lemma 5

Let M be an upper bound for $|f''|$ on (α, ∞) and let $\varepsilon, l \in \mathbb{R}^+$. Let $m_1 \in \mathbb{R}^+$ be such that on any $I \subset (\alpha, \infty)$ of length at least l, $|f'| > m_1$ except on exceptional sets, $S_1, \ldots, S_N$, of relative total length less than ε. We show that there exists an $m = m(\varepsilon, l) > 0$ such that $|f| > m$ on I except on exceptional sets of relative total length less than ε. We may suppose that $m_1 < \varepsilon$.

Let I_1 be one of the intervals making up $I \setminus \cup_{j=1}^N S_j$ and let $x_0 \in I_1$. By Lemma 4, $|f| > m_1^3$ on the interval $(x_0 - m_1/(2M), x_0 + m_1/(2M))$ except perhaps on a sub-interval of length at most $8m_1^2$. Such an interval has relative length at most $8Mm_1$. Hence $|f| > m_1^3$ on I except on a finite set of sub-intervals, $\{T_1, \ldots, T_r\}$, of total length less than $(\varepsilon + 8Mm_1)|I|$. On using the inequality $m_1 < \varepsilon$ and replacing ε by $\varepsilon/(8M+1)$, we obtain that

$$|f(x)| > m = \bar{m_1}^3 \quad \forall x \in I \setminus \cup_{j=1}^r T_j, \tag{5}$$

where $\bar{m_1} = m_1(\varepsilon/(8M+1))$ and $\sum_{j=1}^r |T_j| < \varepsilon|I|$.

This completes the proof of Lemma 5.

Lemma **6** *Suppose that $f \in \mathcal{S}$ and that b is an element of a Hardy field such that $b(x)$ tends to infinity and $b'(x)$ is eventually increasing. Then $f \circ b \in \mathcal{S}$.*

Proof of Lemma 6

By adjusting a, we may supppose that $b'(x)$ is everywhere increasing. Then the functional inverse $h(x) = b^{-1}$ exists, belongs to a Hardy field, tends to infinity and its derivative is decreasing. Since $h(x) \to \infty$, $h(x) > x^{-\delta}$ in the Hardy–field ordering for every $\delta \in \mathbb{R}^+$, and hence $h'(x) > x^{-1-\delta}$. Thus for $c, d \in \mathbb{R}$ with $c < d$, we have

$$\log\left(\frac{h'(d)}{h'(c)}\right) = \int_c^d (\log h'(t))' dt \geq (-1-\delta) \int_c^d (\log t)' dt = (-1-\delta)\log(d/c).$$

Hence

$$h'(d) > h'(c)(c/d)^{1+\delta} \tag{6}$$

Now let ε and l be given elements of $\mathbb{R}^+$, and suppose that $f \in \mathcal{S}$. On applying the definition with I replaced by $b(I)$ and l replaced by $b(l)$, we see that there exist $a_1, m, M \in \mathbb{R}^+$ such that for every $b(I) \subset (a_1, +\infty)$ of length at least $b(l)$, there exist $S_1, \ldots, S_N \subset b(I)$ with $m < |f| < M$ on $b(I) \setminus \cup_{j=1}^N S_j$ and $\sum_{j=1}^N |S_j| < \varepsilon|b(I)|$. Let $a = h(a_1)$, so that $a_1 = b(a)$. It follows at once that for $I \subset (a, \infty)$ of length at least l, $m < |f \circ b| < M$ on $I \setminus \cup_{j=1}^N h(S_j)$, and it is a matter of showing that $\sum_{j=1}^N |h(S_j)|$ is suitably small in comparison with $\varepsilon|I|$.

We may replace $b(a)$ by $\max\{b(a), 1\}$ and $b(l)$ by $\min\{b(l), 1/2\}$. Then we may chop up $b(I)$ into pieces of length no more than 1 and at least $b(l)$, and prove the result for each piece separately. On such an interval (c, d) we have $c/d \geq 1/2$ and hence taking $\delta = 1$ in (6) gives $h'(d) > h'(c)/4$. Given I, let us suppose that c and d have been chosen so that $b(I) = (c, d)$, and let $S_j = (c_j, d_j)$. We have for $j = 1, \ldots, N$,

$$\frac{|h(S_j)|}{|I|} = \frac{\int_{c_j}^{d_j} h'}{\int_c^d h'} \leq \frac{h'(c_j)|S_j|}{h'(d)|I|} \leq \frac{h'(c)|S_j|}{h'(d)|I|} \leq 4\frac{|S_j|}{|I|}.$$

Hence

$$\frac{\sum_{j=1}^{N} |h(S_j)|}{|I|} \leq 4\frac{\sum_{j=1}^{N} |S_j|}{|I|} < 4\varepsilon.$$

Thus $f \circ b \in \mathcal{S}$ and this is sufficient to establish Lemma 6.

The analogue of Lemma 6 for the case when b' decreases is as follows.

LEMMA **7** *Suppose that for all $v \in \mathbb{R}$, $f - v \in \mathcal{S}$ and that b is an element of a Hardy field such that $b(x)$ tends to infinity and $b'(x)$ is eventually decreasing. Then $f \circ b \in \mathcal{W} \cup \mathbb{R}$.*

Proof of Lemma 7

If $f - v = 0$ for some $v \in \mathbb{R}$ then $f \in \mathbb{R}$. Thus we may confine our attention to the case when $f - v$ is not zero for any $v \in \mathbb{R}$. Moreover by replacing f by $f + v$, we may take $v = 0$.
Let $\varepsilon \in (0, \frac{1}{4}) \subset \mathbb{R}$ and let $l \in \mathbb{R}^+$ as in the definition of $\mathcal{S}$. Write $I = (a, a + l)$ and $I' = (a, a + 2l)$. Let $S_1,\ldots,S_K$ be the exceptional sets of I', so that $\sum_{j=1}^{K} |S_j| < \varepsilon|I'|$ and $m < |f| < M$ on $I' \setminus \cup_{j=1}^{K} S_j$.
For an interval J with $I \subset J \subset I'$ (or a finite union of such intervals) and a point $p \in J$, we say that property $P(J, p)$ holds if

$$\sum_{j=1}^{N} |S_j \cap (p, a + 2l)| \leq 2\varepsilon|J \cap (p, a + 2l)|.$$

Since $\sum |S_j| < \varepsilon|I'|$ and $a + l$ is the mid-point of I', it is clear that $P(I', p)$ holds for all $p < a + l$ We would like to find a J such that $P(J, p)$ holds for all $p \in J$. If this is not the case for I' itself, let q be the smallest value for which $P(I', q)$ fails, and take $J = (a, q]$. Then $q \geq a + l$ and we claim that $P(J, p)$ holds for all $p \in J$. For if there exists $p < q$ for which $P(J, p)$ fails then

$$\sum_{j=1}^{N} |S_j \cap [p, q]| > 2\varepsilon|J \cap [p, q]| = 2\varepsilon|I' \cap [p, q]|.$$

However since $P(I', q)$ fails, we already have

$$\sum_{j=1}^{N} |S_j \cap [q, a + 2l]| > 2\varepsilon|I' \cap [q, a + 2l]|,$$

and hence

$$\sum_{j=1}^{N} |S_j \cap [p, a + 2l]| > 2\varepsilon|I' \cap [p, a + 2l]|,$$

contrary to the definition of q.
Now with this J we define for $j = N, N-1,\ldots,1$ a set of sub-intervals, $J_{1,j},\ldots,J_{s(j),j}$, contained in $J \setminus \cup_{j=1}^{N} S_j$ and each to the right of S_j such that

(i) The $J_{i,j}$ are pairwise disjoint for $i = 1,\ldots,s(j), j = 1,\ldots,N$.
(ii)

$$|S_j| = \frac{2\varepsilon}{1 - 2\varepsilon} \sum_{i=1}^{s(j)} |J_{i,j}|.$$

Since P holds at the left-hand end-point of S_N, we can do this for S_N alone just by taking $s(N) = 1$ and $J_{1,N}$ to be an interval of the correct length to the right of S_N.
Now suppose that we have the required $J_{i,j}$ for $j = N,\ldots,r+1$. We call a sub-interval of J 'good' if it does not meet $\cup_{j=1}^N S_j$. So J is the union of the good intervals and the exceptional sets. If we remove $S_N,,\ldots,S_{r+1}$ and $J_{i,j}, i = 1,\ldots,s(j), j = r+1,\ldots,N$, from J then the property P remains true of the union of the remaining intervals. Hence $|S_r|$ is less than or equal to $\frac{2\varepsilon}{1-2\varepsilon}$ times the lengths of the remaining good intervals to the right of S_r. So we have sufficient good intervals to the right of S_r to define $J_{1,r},\ldots,J_{s(r),r}$ as required.
Now let j be between 1 and N, let $S_j = (c_j, d_j)$ and let $J_{i,j} = (\alpha_{i,j}, \beta_{i,j})$. Since h' is increasing (where $h = b^{-1}$), then

$$|h(S_j)| = \int_{c_j}^{d_j} h'(t)dt \le (d_j - c_j)h'(d_j)$$

and for each $i = 1,\ldots,s(j)$

$$|h(J_{i,j})| = \int_{\alpha_{i,j}}^{\beta_{i,j}} h'(t)dt \ge (\beta_{i,j} - \alpha_{i,j})h'(\alpha_{i,j}) \ge (\beta_{i,j} - \alpha_{i,j})h'(d_j).$$

Hence

$$\sum_{i=1}^{s(j)} |h(J_{i,j})| \ge h'(d_j)\sum_{i=1}^{s(j)}(\beta_{i,j} - \alpha_{i,j}) = h'(d_j)\frac{(1-2\varepsilon)}{2\varepsilon}(d_j - c_j) \ge \frac{(1-2\varepsilon)}{2\varepsilon}|h(S_j)|.$$

Hence

$$\sum_{j=1}^{N} |h(S_j)| < 4\varepsilon|h(J)|.$$

Moreover $m < |f(b(x))| < M$ on $h(J) \setminus \cup_{j=1}^N h(S_j)$ because $m < |f(y)| < M$ for $y \in b(h(J) \setminus \cup_{j=1}^N h(S_j)) = J \setminus \cup_{j=1}^N S_j$.
Now take $\delta \in \mathbb{R}^+$ such that $\delta < m$ and $\delta^{-1} > M$ and let $h(J) = (\alpha, T)$. Then

$$\mu((\{\delta < |f \circ b| < \delta^{-1}\} \cap (\alpha, T)) \setminus P_{f\circ b}) > (1 - 4\varepsilon)\mu((\alpha, T)).$$

Hence

$$\lim_{a\to\infty}\lim_{\delta\to 0}\text{limsup}_{T\to\infty}\frac{\mu((\{\delta < |f\circ b - v| < \delta^{-1}\} \cap [a,T]) \setminus P_{f\circ b}))}{\mu([a,T])} = 1,$$

and since we may replace f by $f - v$ for any $v \in \mathbb{R}$, this completes the proof of Lemma 7.

To see that we do not necessarily have $f \circ b \in \mathcal{S}$ under the hypotheses of Lemma 7, consider the function $\sin(\log_2 x)$. With ε small, $|\sin(\log_2 x)| < \varepsilon$ if $\pi N - \varepsilon < \log_2 x < \pi N$, $N \in \mathbb{N}$, i.e. if $\exp_2(\pi N - \varepsilon) < x < \exp_2(\pi N)$. Relative to $(0, \exp_2(\pi N))$, this interval has length

$$\frac{\exp_2(\pi N) - \exp_2(\pi N - \varepsilon)}{\exp_2(\pi N)} = 1 - \exp(e^{\pi N}(e^{-\varepsilon} - 1)) \sim 1 - e^{-\varepsilon e^{\pi N}} \to 1,$$

as $N \to \infty$. So $\sin(\log_2 x) \notin \mathcal{S}$, although by Lemma 7 this function does belong to $\mathcal{W} \cup \mathbb{R}$. This is the reason why we need the limsup in the definition of $\mathcal{W}$. Note also that our need to proceed via $\mathcal{S}$ comes from the fact that $\mathcal{W}$ is not closed under composition with Hardy–field elements.

Our main result of this section is the following.

THEOREM **1**

$$\mathcal{R} \subset \mathcal{W} \cup \mathbb{R}.$$

Proof of Theorem 1
We shall first prove, by induction on k, that if $x/b_1(x)$ tends to a finite limit and $f \in \mathcal{R}$, then $f \in \mathcal{S}$. Lemmas 6 and 7 will then give the conclusion of the theorem. The induction step is obtained from an argument involving the Wronskian and Lemmas 4 and 6. The base case $k = 0$ is trivial.
So suppose that $x/b_1(x)$ tends to a finite limit, and let

$$f \in \mathbb{R}[\sin(\lambda_{1,1}b_1(x) + d_{1,1}),\ldots, \sin(\lambda_{1,J_1}b_1(x) + d_{1,J_1}),\ldots, \sin(\lambda_{k,J_k}b_k(x) + d_{k,J_k})].$$

For convenience, we write $\lambda_{1,i} = \lambda_i$, $i = 1,\ldots,J_1$.
By Lemma 3, f may be written in the form

$$f(x) = P_0(x) + \sum_{i=1}^{s}\{P_i(x)\sin(\lambda_i b_1(x)) + Q_i(x)\cos(\nu_i b_1(x))\},$$

where each P_i and Q_i belongs to

$$\mathbb{R}[\sin(\lambda_{2,1}b_2(x)), \cos(\nu_{2,1}b_2(x)),\ldots, \cos(\nu_{2,J_1}b_2(x)),\ldots, \sin(\lambda_{k,J_k}b_k(x)), \cos(\nu_{k,J_k}b_k(x))].$$

We may suppose that b_1 is everywhere increasing in the intervals we shall consider in which case it has a well defined inverse function. We write $\tilde{f} = f \circ b_1^{-1}$. In a similar vein, we write $\tilde{P}_0 = P_0 \circ b_1^{-1}$ and so on. Then

$$\tilde{f}(x) = \tilde{P}_0(x) + \sum_{i=1}^{s}\{\tilde{P}_i(x)\sin(\lambda_i x) + \tilde{Q}_i(x)\cos(\nu_i x)\}. \tag{7}$$

We may assume that the λ_i are pairwise distinct, and likewise the ν_i. Then the Wronskian $\tilde{W} = W(\tilde{f}, \tilde{f}',\ldots,\tilde{f}^{2s})$ is a polynomial in the $\tilde{P}_i$ and $\tilde{Q}_i$; see [41] for example. By induction on k, $W = \tilde{W} \circ b_1$ belongs to $\mathcal{S}$. So given $\varepsilon, l \in \mathbb{R}^+$ there exists an $m = m(\varepsilon, l) > 0$ and an $a \in \mathbb{R}$ such that for any interval $I \subset (a, \infty)$ of length at least l there are exceptional sets S_i, $i = 1,\ldots,N$ with $\sum |S_i| < \varepsilon|I|$ and $\tilde{W} > m$ on $I \setminus \cup_1^N S_i$.
It follows that there exists an $m_1 = m_1(\varepsilon, l) > 0$ such that for each $x \in I \setminus \cup_1^N S_i$ there is a $j = j(x)$ with $0 \leq j \leq 2s$ and $\tilde{f}^{(j)}(x) > m_1$. Essentially this is because not all the derivatives can be small or the Wronskian would have to be small. Now $\tilde{f}^{(2s+1)}$ is bounded, say by M, and so by Lemma 4 any point where $|\tilde{f}^{(2s)}| > m_1$ can be included in an interval where $|\tilde{f}^{(2s-1)}| > m_1^3$ except on exceptional sets of relative length $8m_1M$. So by excluding these sets we can cover I with interval where one of $|\tilde{f}|, |\tilde{f}'|,\ldots,|\tilde{f}^{(2s-1)}|$ is greater then m_1^3. But we can repeat the above with $2s$ replaced by $2s - 1$, and by continuing in this way we see that there is an $m = m(\varepsilon, l) > 0$ such that $|\tilde{f}| > m$ on I except on exceptional sets of relative length no more that ε. Thus $\tilde{f} \in \mathcal{S}$ and Lemma 6 gives that $f = \tilde{f} \circ b_1 \in \mathcal{S}$. Clearly the same applies to $f - v$ for any $v \in \mathbb{R}$.
For a general b_1 we have that $f \circ b_1^{-1} - v \in \mathcal{S}$ for all $v \in \mathbb{R}$. If b_1' is eventually increasing then Lemma 6 gives that $f - v \in \mathcal{S}$, and the argument at the end of the proof of Lemma 7 shows that $f \in \mathcal{W} \cup \mathbb{R}$. If b_1' is eventually decreasing then Lemma 7 itself gives that $f \in \mathcal{W} \cup \mathbb{R}$. Since $\mathcal{W} \cup \mathbb{R}$ is closed under division, it follows that $\mathcal{R} \subset \mathcal{W} \cup \mathbb{R}$ as required.

3 Multiseries

Classically the asymptotic growth of a function has often been expressed by giving an asymptotic power series expansion, [16, 10]. However the growth of $f(x)$ cannot always be expressed by powers of x. For example one may need to use exponentials and logarithms as well. Thus

$$\log\{x + e^{-x}\} = \log x + \frac{e^{-x}}{x} - \frac{e^{-2x}}{2x^2} + \cdots. \tag{8}$$

Therefore when we want to expand an exp–log function, say, the first need is for a *scale*. Essentially this is a finite set of functions whose powers are of non–comparable asymptotic growth; for example $\{\log x, x, e^x\}$ in (8). However for algorithmic purposes, we need to be able to construct complicated scale elements from simpler ones, and so our definition recurses with that of a multiseries, [37, 28]. It is convenient here to use scale elements which tend to zero.

DEFINITION **1** *Let $\mathcal{F}$ be a Hardy field and let $\mathbf{t_1}, \ldots, \mathbf{t_n}$ be elements of $\mathcal{F}$ which tend to zero and satisfy $\log \mathbf{t_i} = \mathbf{o}(\log \mathbf{t_{i+1}})$ for $i = 1,\ldots,n-1$. We say that $\{\mathbf{t_1}, \ldots, \mathbf{t_n}\}$ is an* asymptotic scale *if the following properties hold:*

1. *$x^{-1} \in \{\mathbf{t_1}, \ldots, \mathbf{t_n}\}$.*
2. *Each $\mathbf{t_i}$ is either of the form $\log_k^{-1} x$ or else $\log \mathbf{t_i}$ has a multiseries expansion in the scale $\{\mathbf{t_1}, \ldots, \mathbf{t_{i-1}}\}$ with every term in the $\mathbf{t_{i-1}}$ expansion tending to plus or minus infinity.*
3. *If $\log_k^{-1} x$ belongs to $\{\mathbf{t_1}, \ldots, \mathbf{t_n}\}$ for some $k > 0$ then so do $\log^{-1} x,\ldots, \log_{k-1}^{-1} x$.*

A multiseries with a one–element scale is just an asymptotic series with non–integral powers allowed.

DEFINITION **2** *We say that an element g of $\mathcal{F}$ has an* asymptotic $\mathbf{t_1}$-expansion $\sum c_m \mathbf{t_1^{r_m}}$ *if $\{c_m\}$ and $\{r_m\}$ are sequences of real numbers, with r_m strictly increasing to infinity, such that for each $N \geq 0$ there is a strictly positive real number, δ_N,with*

$$g - \sum_{m=0}^{N} c_m \mathbf{t_1^{r_m}} = \mathrm{O}(\mathbf{t_1^{r_N+\delta_N}}). \tag{9}$$

The general definition is as follows.

DEFINITION **3** *Let $f \in \mathcal{F}$, and suppose that there exists a strictly increasing sequence of real numbers, $\{r_m\}$ with $r_m \to \infty$, and a sequence of elements $\{g_m\} \subset \mathcal{F}$ such that for each $N \geq 0$ there is a positive real number δ_N with*

$$g - \sum_{m=0}^{N} g_m \mathbf{t_n^{r_m}} = \mathrm{O}(\mathbf{t_n^{r_N+\delta_N}}). \tag{10}$$

Then we say that g has $\{\mathbf{t_1}, \ldots, \mathbf{t_n}\}$ multiseries expansion $\sum g_m \mathbf{t_n^m}$ *provided that each g_m has a $\{\mathbf{t_1}, \ldots, \mathbf{t_{n-1}}\}$ multiseries expansion.*

In practice each g_m needs to be in closed form so that zero–equivalence testing is possible; see below. We shall refer to (10) as the $\mathbf{t_n}$-*expansion* of g. It is not hard to see that both scales and multiseries are well defined by the recursion. Multiseries are essentially an algorithmic version of the transseries of Ecalle, [12, 13], although their development proceeded independently.

We also need standard classes of input function, as well as standard ways of expressing growth. An idea from differential algebra is that of a *tower of function fields*. For us this is a finite sequence of fields of functions

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m,$$

with each $\mathcal{F}_i$ a simple extension of $\mathcal{F}_{i-1}$, so that $\mathcal{F}_i = \mathcal{F}_{i-1}(f_i)$, $i = 1,\ldots,m$. Often $\mathcal{F}_0$ is a field of constants. When each f_i is an exponential or a logarithm of an element of $\mathcal{F}_{i-1}$ the $\mathcal{F}_m$ is a field of exp–log functions.
Other function classes may be treated by allowing different sorts of f_i. There are several requirements at each stage if this is going to work.

1. We have to be able to decide whether the inclusion of f_i necessitates a new scale element and to expand f_i in the (new) scale.

2. We have to be able to decide zero equivalence in $\mathcal{F}_{i-1}(f_i)$.

3. We have to avoid all problems of indefinite cancellation, which would arise if we were to subtract two series with identical tails term by term.

The second requirement is highly non–trivial. In fact there is no known algorithm to decide the zero equivalence of exp–log constants. Algorithms based on conjectures are known, [20, 39, 40, 30]. Outside the scope of these, it is necessary to postulate the existence of an oracle to decide the sign of a constant. Given a method for constants, there are a number of algorithms for functions, [34, 18], although there can be problems of space and time in implementation.

4 An Algorithm For Expansions With Trigonometric Coefficients

Let $\mathbb{K}$ be a field of real constants. We write $\mathcal{E}_R$ for the field of functions given by expressions generated by the signature $\mathbb{K}, x, +, -, \times, \div, \exp, \log, \sin$ subject to the restriction that sin may not appear in any sub-expression which is an argument of any of $\exp, \log, \sin$. The main aim of this section is to give an algorithm to compute a multiseries expansion of a given element of $\mathcal{E}_R$ with coefficients in $\mathcal{R}$.
Let $\mathcal{F}$ denote the field of functions given by the signature $\mathbb{Q}, , x, +, -, \times, \div, \exp, \log, \sin$ subject now to the restriction that any argument of a sine must tend to a finite limit. In [35] it is shown that $\mathcal{F}$ is an asymptotic field. This means in particular that it is a Hardy field and that modulo zero–equivalence of constants we can compute multiseries for elements of $\mathcal{F}$.
The essence of our present algorithm is as follows. Given an expression, E, we build a scale for all the exp–log sub–expressions of E (that is to say those sub–expressions which do not contain any trigonometric functions). We then use the exp–log algorithm to split the

argument, a, of each trigonometric function into a part a_∞ whose modulus tends to infinity, a constant, a_c and a part which tends to zero, a_z, so that $a = a_\infty + a_c + a_z$. Of course a_∞ will be absent if a has a finite limit. It is clear that the splitting can be done as follows:

> We compute the terms of the multiseries with respect to the most rapidly varying scale element until we reach a term with a negative exponent. Any terms with positive exponent go in a_∞, the tail consisting of the terms with negative exponent goes in a_z and if there is a term with zero exponent its coefficient is similarly expanded with respect to the next scale element, and so on.

We then use the addition formula to split $\sin a$ into trigonometric functions of a_∞, a_c and a_z. Finally we use the algorithm of [35] to compute a multiseries for E with the various $\sin a_\infty$, $\cos a_\infty$, $\sin a_c$ and $\cos a_c$ being regarded as constants and the $\sin a_z$ and $\cos a_z$ as elements of $\mathcal{F}$ above.

We now give the algorithm in more detail. Suppose that we have a function tower

$$\mathbb{Q} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$$

where either $\mathcal{F}_i = \mathcal{F}_{i-1}(f_i)$, $1 \le i \le n$, with f_i an exponential or a logarithm of an element, g_i of $\mathcal{F}_{i-1}$, or $\mathcal{F}_i = \mathcal{F}_{i-1}(\sin g_i, \cos g_i)$. In all cases there is the restriction that g_i contains no sines in its expression. With each $\mathcal{F}_i$ we associate a scale $\mathcal{T}(\mathcal{F}_i)$, an argument list $\mathcal{A}(\mathcal{F}_i)$, a coefficient field $\mathcal{K}(\mathcal{F}_i)$ and a set of *z–functions* $\mathcal{Z}(\mathcal{F}_i)$. The argument list $\mathcal{A}(\mathcal{F}_i)$ contains the arguments, to within a constant multiple, of sines that make up $\mathcal{K}(\mathcal{F}_i)$. The z–functions are given by expressions of one of the forms, $\exp z, \log(1+z), \sin z, \cos z, (1+z)^c$, where $c \in \mathbb{R} \setminus \mathbb{N}$ and z is an element of $\mathcal{F}_i$ which tends to zero.
Once the tower with its associated data structures is in place, the main step of the algorithm is to write the given expression $F \in \mathcal{F}_n$ as a polynomial expression in real powers of scale elements and z–functions with coefficients in $\mathcal{K}(\mathcal{F}_n)$. Standard power series expansions may then be used to obtain a multiseries for F.

Algorithm *Let F be a function given by an expression built from the integers and the variable x using arithmetic operations and the functions exp, log and sin, subject to the restriction that sin may not appear in any subexpression of an argument to exp, log or sin itself.*
We give a method for computing a multiseries of F.

1. *Construct a tower of functions, $\mathbb{Q} \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ as above with $F \in \mathcal{F}_n$.*

2. *For $\mathcal{F}_0 = \mathbb{Q}$, set $\mathcal{T}(\mathcal{F}_0) = \emptyset$, $\mathcal{A}(\mathcal{F}_0) = \emptyset$, $\mathcal{K}(\mathcal{F}_0) = \mathbb{Q}$ and $\mathcal{Z}(\mathcal{F}_0) = \emptyset$.*

3. *Take $\mathcal{F}_1 = \mathbb{Q}(x)$, $\mathcal{T}(\mathcal{F}_1) = \{x^{-1}\}$, $\mathcal{A}(\mathcal{F}_1) = \emptyset$, $\mathcal{K}(\mathcal{F}_1) = \mathbb{Q}$ and $\mathcal{Z}(\mathcal{F}_0) = \emptyset$.*

4. *Assuming that $\mathcal{T}(\mathcal{F}_{i-1})$, $\mathcal{A}(\mathcal{F}_{i-1})$, $\mathcal{K}(\mathcal{F}_{i-1})$ and $\mathcal{Z}(\mathcal{F}_{i-1})$ have been fixed ($i = 2,\ldots,n$), we define the corresponding quantities for $\mathcal{F}_i$ as follows.*

5. *Suppose that $f_i = \exp g_i$, with $g_i \in \mathcal{F}_{i-1}$.*

   (a) *Then we calculate the multiseries of g_i sufficiently to be able to split g_i as $g_i = g_\infty + g_c + g_z$ where $g_\infty \to \infty$, g_c is constant and $g_z \to 0$.*

(b) *For each* $\mathbf{t_j} \in \mathcal{T}(\mathcal{F}_{\mathbf{i-1}})$ *we calculate the limit of* $g_\infty/\log \mathbf{t_j}$*. If it is finite and non–zero, say equal to* k*, we replace* g_i *by* $g_i - k\log \mathbf{t_j}$*,* f_i *by* $f_i\mathbf{t_j^{-k}}$ *and if this reduces* g_∞ *to zero we are done with this stage. Otherwise we continue comparing the new* g_∞ *with logs of scale elements. Note that we can do these computations by calculating a sufficient number of terms of the multiseries of* g_i*.*

(c) *If the limit of* $g_\infty/\log \mathbf{t_j}$ *is infinite or zero for each* j*, we add* $e^{\pm g_\infty}$ *as a new scale element. So* $\mathcal{T}(\mathcal{F}_i) = \mathcal{T}(\mathcal{F}_{i-1}) \cup \{e^{\pm g_\infty}\}$*, with the + or − sign taken to ensure that the new scale element tends to zero.*

(d) *We set* $\mathcal{A}(\mathcal{F}_i) = \mathcal{A}(\mathcal{F}_{i-1})$*,* $\mathcal{K}(\mathcal{F}_i) = \mathcal{K}(\mathcal{F}_{i-1})(e^{g_c})$ *and* $\mathcal{Z}(\mathcal{F}_i) = \mathcal{Z}(\mathcal{F}_{i-1})(e^{g_z})$*.*

Note that we have a polynomial expression for f_i *in terms of powers of scale elements and z–functions.*

6. *Now suppose that* $f_i = \log g_i$ *with* $g_i \in \mathcal{F}_{i-1}$*. By considering the first term in the* $\mathbf{t_n}$*–expansion of* g_i*, and the first term in its expansion, and so on, we can compute an expression for* g_i *of the form*

$$g_i = A\mathbf{t_n^{\nu_n}}\mathbf{t_{n-1}^{\nu_{n-1}}},\ldots,\mathbf{t_1^{\nu_1}}(\mathbf{1}+\epsilon), \tag{11}$$

where A *is a non–zero constant and* ϵ *tends to zero and has a computable* $\{\mathbf{t_1}\ldots,\mathbf{t_n}\}$ *multiseries expansion. Then*

$$\log g_i = \nu_n \log \mathbf{t_n} + \nu_{\mathbf{n-1}} \log \mathbf{t_{n-1}} + \cdots + \nu_{\mathbf{1}} \log \mathbf{t_1} + \log \mathbf{A} + \log(\mathbf{1}+\epsilon). \tag{12}$$

Note that $\mathbf{t_1}$ *will be of the form* $(\log_k x)^{-1}$ *for some* k*. Unless* $\nu_1 = 0$*, we must add* $-(\log \mathbf{t_1})^{-\mathbf{1}}$ *as a new scale element, so* $\mathcal{T}(\mathcal{F}_i) = \mathcal{T}(\mathcal{F}_{i-1}) \cup \{(\log_{k+1} x)^{-1}\}$*. Similarly* $\mathcal{A}(\mathcal{F}_i) = \mathcal{A}(\mathcal{F}_{i-1})$*,* $\mathcal{K}(\mathcal{F}_i) = \mathcal{K}(\mathcal{F}_{i-1})(\log A)$ *and* $\mathcal{Z}(\mathcal{F}_i) = \mathcal{Z}(\mathcal{F}_{i-1}) \cup \{\log(1+\epsilon)\}$*.*

Again we have a polynomial expression for f_i *in terms of powers of scale elements and the z–function* $\log(1+\epsilon)$*.*

7. *Finally we consider the case when* $f_i = \sin g_i$*.*

(a) *For each* $b_j \in \mathcal{A}(\mathcal{F}_{i-1})$ *we compute the limit of* g_i/b_j*. If this is finite and non–zero, say equal to* l*, we replace* g_i *by* $g_i - lb_j$ *and* f_i *by* $\sin(g_i - lb_j)$*. We then repeat the computation with any smaller remaining elements of* $\mathcal{A}(\mathcal{F}_{i-1})$*.*

(b) *We split* g_i *as* $g_i = g_\infty + g_c + g_z$ *as above. Then we use the addition formulae to write* $\sin g_i$ *in terms of sines and cosines of* g_∞*,* g_c *and* g_z*.*

(c) *Assuming that* g_∞ *and* g_c *are present, we add their sines and cosines to the coefficient field, so that*

$$\mathcal{K}(\mathcal{F}_i) = \mathcal{K}(\mathcal{F}_{i-1})(\sin g_\infty, \cos g_\infty, \sin g_c, \cos g_c).$$

(d) *We set* $\mathcal{A}(\mathcal{F}_i) = \mathcal{A}(\mathcal{F}_{i-1}) \cup \{g_\infty\}$ *and* $\mathcal{T}(\mathcal{F}_i) = \mathcal{T}(\mathcal{F}_{i-1})$*.*

(e) *Similarly we set* $\mathcal{Z}(\mathcal{F}_i) = \mathcal{Z}(\mathcal{F}_{i-1}) \cup \{\sin g_z, \cos g_z\}$*, and note that once again we have a polynomial expression for* f_i *in terms of z–functions using our new coefficient set.*

8. *We can now rewrite our input expression F as a polynomial in powers of scale elements and z–functions with coefficients in $\mathcal{K}(\mathcal{F}_n)$. All that remains is to expand the z–functions, but there are two cautions. Firstly we must expand with respect to the $\mathbf{t_n}$ first, and then the coefficients with respect to $\mathbf{t_{n-1}}$, and so on. Secondly, if the $\mathbf{t_n}$–expansion of a z–function begins with a term in $\mathbf{t_n^0}$, we need to use the functional equation before expanding. Thus if b_j has $\mathbf{t_n}$–expansion $b_j(x) = \sum_{m=0}^{\infty} p_m \mathbf{t_n^{r_m}}$ with $r_0 = 0$, then for example we write*

$$\sin b_j = \sin(p_0)\cos\left(\sum_{m=1}^{\infty} p_m \mathbf{t_n^{r_m}}\right) + \cos(p_0)\sin\left(\sum_{m=1}^{\infty} p_m \mathbf{t_n^{r_m}}\right),$$

and then expand the terms on the right. If we were to expand directly, the coefficient of say $\mathbf{t_n}$ might have to be extracted from infinitely many terms of the series

$$\sin b_j = \sum_{k=1}^{\infty} \frac{(p_0 + \sum_{m=1}^{\infty} p_m \mathbf{t_n^{r_m}})^{\mathbf{2k-1}}}{(2k-1)!}.$$

4.1 Examples

Our first example is a simple one, in that only powers of x are involved in the expansion. Let F be given by the expression

$$F(x) = x^2 \sin\left(x + \frac{1}{x}\right) + x\cos\left(x - \frac{1}{x^2}\right) + \frac{1}{x^3}.$$

Here the function tower is

$$\mathcal{F}_0 = \mathbb{Q} \subset \mathcal{F}_1 = \mathbb{Q}(x) \subset \mathcal{F}_2 = \mathbb{Q}\left(x, \sin\left(x + \frac{1}{x}\right), \cos\left(x + \frac{1}{x}\right)\right) \subset$$
$$\mathcal{F}_3 = \mathbb{Q}\left(x, \sin\left(x + \frac{1}{x}\right), \cos\left(x + \frac{1}{x}\right), \sin\left(x - \frac{1}{x^2}\right), \cos\left(x - \frac{1}{x^2}\right)\right).$$

We have $\mathcal{T}(\mathcal{F}_1) = \{x^{-1}\}$, $\mathcal{A}(\mathcal{F}_1) = \emptyset$, $\mathcal{K}(\mathcal{F}_1) = \mathbb{Q}$ and $\mathcal{Z}(\mathcal{F}_1) = \emptyset$. At the next stage we have to add $\sin(x + 1/x)$ and $\cos(x + 1/x)$. Now $g_2 = x + 1/x$ and we see that $g_\infty = x$, $g_c = 0$ and $g_z = x^{-1}$. So $\mathcal{T}(\mathcal{F}_2) = \mathcal{T}(\mathcal{F}_1)$, $\mathcal{A}(\mathcal{F}_2) = \{x\}$, $\mathcal{K}(\mathcal{F}_2) = \mathbb{Q}(\sin x, \cos x)$ and $\mathcal{Z}(\mathcal{F}_2) = \{\sin(x^{-1}), \cos(x^{-1})\}$. We note that $f_2 = \sin(x + x^{-1})$ can be written in terms of the coefficients and $\sin(x^{-1}), \cos(x^{-1})$.
Finally we have to add $\sin(x - x^{-2})$ and $\cos(x - x^{-2})$. We discover that $g_3 = x - x^{-2}$ is asymptotic to our element of $\mathcal{A}(\mathcal{F}_2)$, and so we replace g_3 by $g_3 - x = -x^{-2}$. Since this tends to zero, we do not need to increase the coefficient field and we have two new z–functions, namely $\sin(x^{-2})$ and $\cos(x^{-2})$; so $\mathcal{Z}(\mathcal{F}_3) = \{\sin(x^{-1}), \cos(x^{-1}), \sin(x^{-2}), \cos(x^{-2})\}$. We write F as a polynomial in the base functions with coefficients in $\mathbb{Q}(\sin x, \cos x)$ and expand.

$$\begin{aligned} F(x) &= x^2 \sin x \cos(x^{-1}) + x^2 \cos x \sin(x^{-1}) + x\cos x \cos(x^{-2}) + x \sin x \sin(x^{-2}) + x^{-3} \\ &= x^2 \left\{\sin x\left(1 - \frac{1}{2!}x^{-2} + \frac{1}{4!}x^{-4} - \frac{1}{6!}x^{-6} + \cdots\right) + \cos x\left(x^{-1} - \frac{1}{3!}x^{-3} + \frac{1}{5!}x^{-5} + \cdots\right)\right\} \\ &\quad + x\left\{\cos x\left(1 - \frac{1}{2!}x^{-4} + \frac{1}{4!}x^{-8} + \cdots\right) + \sin x\left(x^{-2} - \frac{1}{3!}x^{-6} + \cdots\right)\right\} \\ &= x^2 \sin x + 2x\cos x - \frac{\sin x}{2} + x^{-1}\left(\sin x - \frac{\cos x}{3}\right) + x^{-2}\frac{\sin x}{24} + x^{-3}\left(1 + \frac{59\cos x}{120}\right) \\ &\quad - x^{-4}\frac{\sin x}{720} - x^{-5}\left(\frac{\sin x}{6} + \frac{\cos x}{5040}\right) + \cdots \end{aligned}$$

Note that the coefficients of the powers of x are all in $\mathbb{Q}(\sin x, \cos x)$.

The second example involves two comparability classes and a less trivial expansion is needed to determine a_∞, a_c and a_z when a is the argument to the sine. Let

$$G(x) = x^2 \sin\left(\frac{e^{2x}+x}{e^x - x}\right) - x\cos\left(\frac{e^x+1}{e^x-x}\right).$$

Here the scale is $\{x^{-1}, e^{-x}\}$. A short computation reveals that

$$\frac{e^{2x}+x}{e^x-x} = e^x + x + \frac{x^2+x}{e^x-x}$$

and similarly

$$\frac{e^x+1}{e^x-x} = \frac{e^x}{e^x-x} + \frac{1}{e^x-x}.$$

Thus

$$\begin{aligned}
G(x) &= x^2\left\{\sin(e^x+x)\cos\left(\frac{e^{-x}(x^2+x)}{1-xe^{-x}}\right) + \cos(e^x+x)\sin\left(\frac{e^{-x}(x^2+x)}{1-xe^{-x}}\right)\right\} \\
&\quad -x\left\{\cos\left(\frac{e^x}{x+1}\right)\cos\left(\frac{1}{x+1}\right) - \sin\left(\frac{e^x}{x+1}\right)\sin\left(\frac{1}{x+1}\right)\right\} \\
&= x^2\sin(e^x+x)\left(1 - \frac{e^{-2x}(x^2+x)^2}{2(1-xe^{-x})^2} + \frac{e^{-4x}(x^2+x)^4}{24(1-xe^{-x})^4} + \cdots\right) \\
&\quad +x^2\cos(e^x+x)\left(\frac{e^{-x}(x^2+x)}{2(1-xe^{-x})^2} - \frac{e^{-3x}(x^2+x)^3}{6(1-xe^{-x})^3} + \cdots\right) \\
&\quad +x\cos\left(\frac{e^x}{x+1}\right)\left(1 - \frac{1}{2(x+1)^2} + \frac{1}{24(x+1)^4} + \cdots\right) - x\sin\left(\frac{e^x}{x+1}\right)\left(\frac{1}{x+1} + \frac{1}{6(x+1)^3} + \cdots\right) \\
&= \left[x^2\sin(e^x+x) + x\cos\left(\frac{e^x}{x+1}\right) - \sin\left(\frac{e^x}{x+1}\right) + x^{-1}\left(\sin\left(\frac{e^x}{x+1}\right) - \frac{1}{2}\cos\left(\frac{e^x}{x+1}\right)\right) + \cdots\right] \\
&\quad +e^{-x}[x^4\cos(e^x+x) + x^3\cos(e^x+x)] - e^{-2x}\sin(e^x+x)\left[\frac{x^6}{2} + x^5 + \frac{x^4}{2}\right] + \cdots
\end{aligned}$$

Here the coefficients of the powers of x are in

$$\mathbb{Q}\left(\sin(e^x+x), \cos(e^x+x), \sin\left(\frac{e^x}{x+1}\right), \cos\left(\frac{e^x}{x+1}\right)\right).$$

5 Appendix - Hardy Fields

We consider $\mathcal{C}^\infty$ functions defined on some interval (c, ∞) and regard two functions as equivalent if they agree on such a set. In other words we are looking at germs at ∞ of $\mathcal{C}^\infty$ functions. This is mainly for convenience since it avoids a lot of unnecessary bookkeeping about precisely where functions are defined.

A *Hardy field* is a field of such functions which is closed under differentiation, [9]. If f is a non–constant element of a Hardy field, then f' has to have a field inverse and so must be eventually positive or eventually negative. Hence f tends to a limit, finite or infinite. Moreover if g is another element of the same Hardy field, $f - g$ must be eventually of

constant sign. The allows us to set $f > g$ if $f - g$ is eventually positive and so obtain a total order on the Hardy field. These properties - the fact that an element tends to a limit and the ability to compare elements according to their asymptotic behaviour - make Hardy fields a powerful tool in the theory of asymptotics.
Examples of Hardy fields include the set of exp–log functions. Indefinite integrals and real roots may also be added to the signature.
Further properties of Hardy fields can be found in [9], [22], [17], [2, 3, 4, 5, 6, 7, 8], [23, 24, 25, 26], [29, 27], [37, 35, 32, 33, 36, 38].

References

[1] R. Bellman. *Stability Theory of Differential Equations.* McGraw-Hill, 1953.

[2] M. Boshernitzan. An extension of Hardy's class L of 'Orders of infinity'. *J. Analyse Math*, 39:235–255, 1981.

[3] M. Boshernitzan. New 'Orders of Infinity'. *J. Analyse Math.*, 41:130–167, 1982.

[4] M. Boshernitzan. Discrete orders of infinity. *Amer. J. Math.*, 106:1147–1198, 1984.

[5] M. Boshernitzan. 'Orders of infinity' generated by difference equations. *Amer. J. Math.*, 106:1067–1089, 1984.

[6] M. Boshernitzan. Hardy fields, existence of transexponential functions and the hypertranscendence of solution to $g(g(x)) = e^x$. *Aequationes Math.*, 30:258–280, 1986.

[7] M. Boshernitzan. Second-order differential equations over Hardy fields. *J. London Math. Soc.*, 35:109–120, 1987.

[8] M. Boshernitzan. Uniform distribution and Hardy fields. *J. D'Analyse Math.*, 62:225–240, 1994.

[9] N. Bourbaki. *Éléments de Mathématiques. Ch. V: Fonctions d'une variable réelle. Appendice, pp. 36–55.* Hermann, Paris, Second edition, 1961.

[10] T.J.I'A Bromwich. *An Introduction to the Theory of Infinite Series, 2nd ed.* Macmillan, 1926.

[11] B.I. Dahn and P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Math.*, 127:45–50, 1986.

[12] J. Ecalle. *Les fonctions résurgentes, 3 vols.* Publ. Math. d'Orsay, Paris, 1981-5.

[13] J. Ecalle. *Introduction aux fontions analysables et preuve constructive de la conjecture de Dulac.* Hermann, Paris, 1992.

[14] D. Gruntz. *On Computing Limits in a Symbolic Manipulation System.* PhD thesis, E.T.H., Zurich., 1996.

[15] P.R. Halmos. *Measure Theory.* Springer-Verlag, New York, Heidelberg, Berlin, 1974.

[16] G.H. Hardy. *Divergent Series.* Oxford Univ. Press, 1949.

[17] A. Lightstone and A. Robinson. *Nonarchimedean Fields and Asymptotic Expansions.* Elsevier, New York, 1975.

[18] Ariane Péladan-Germa. Testing identities of series defined by algebraic partial differential equations. In Gérard Cohen, Marc Giusti, and Teo Mora, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 393–407. Springer-Verlag, 1995. Proceedings of the 11th International Symposium, AAECC-11, Paris, France, July 1995.

[19] D. Richardson. Some undecidable problems involving elementary functions of a real variable. *J. Symbolic Logic*, 33:514–520, 1968.

[20] D. Richardson. The uniformity conjecture. pages 253–272. Springer Verlag, 2001. Lecture Notes in Computer Science.

[21] Dan Richardson, Bruno Salvy, John Shackell, and Joris Van der Hoeven. Asymptotic expansions of exp-log functions. In Y. N. Lakshman, editor, *ISSAC'96*, pages 309–313. ACM Press, 1996. Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation. July 24–26, 1996. Zurich, Switzerland.

[22] A. Robinson. On the real closure of a Hardy field. In G. Asser et al., editor, *Theory of Sets and Topology*, Berlin, 1972. Deut. Verlag Wissenschaften.

[23] M. Rosenlicht. Hardy fields. *J. Math. Anal. App.*, 93/2:297–311, 1983.

[24] M. Rosenlicht. The rank of a Hardy field. *Trans. Amer. Math. Soc.*, 280/2:659–671, 1983.

[25] M. Rosenlicht. Rank change on adjoining real powers to Hardy fields. *Trans. Amer. Math. Soc.*, 284/2:829–836, 1984.

[26] M. Rosenlicht. Growth properties of functions in Hardy fields. *Trans. Amer. Math. Soc.*, 299/1:261–272, 1987.

[27] B. Salvy and J. Shackell. Symbolic asymptotics: Functions of two variables, implicit functions. *Journal of Symbolic Computation*, 25(3):329–349, March 1998.

[28] B. Salvy and J. Shackell. Symbolic asymptotics: Multiseries of inverse functions. *Journal of Symbolic Computation*, 27(6):543–563, June 1999.

[29] B. Salvy and J.R. Shackell. Asymptotic expansions of functional inverses. In *ISSAC92 proceedings, Berkeley, California*, pages 130–137. ACM Press, 1992.

[30] J. Shackell and J. Van Der Hoeven. Complexity bounds for zero–test algorithms. Research report UKC/IMS/01/46, University of Kent, Canterbury, 2001.

[31] J.R. Shackell. Growth estimates for exp-log functions. *J. Symbolic Comp.*, 10/6:611–632, 1990.

[32] J.R. Shackell. Inverses of Hardy L-functions. *Bull. London Math. Soc.*, 25:150–156, 1993.

[33] J.R. Shackell. Rosenlicht fields. *Trans. Amer. Math. Soc.*, 335/2:579–595, 1993.

[34] J.R. Shackell. Zero-equivalence in function fields defined by algebraic differential equations. *Trans. Amer. Math. Soc.*, 336/1:151–172, 1993.

[35] J.R. Shackell. Extensions of asymptotic fields via meromorphic functions. *J. London Math. Soc.*, 52:356–374, 1995.

[36] J.R. Shackell. Growth orders occurring in expansions of Hardy-field solutions of algebraic differential equations. *Annales de l'Institut Fourier*, 45:183–221, 1995.

[37] J.R. Shackell. Limits of Liouvillian functions. *Proc. London Math. Soc.*, 72:124–156, 1996.

[38] J.R. Shackell and B. Salvy. Asymptotic forms and algebraic differential equations. *J. Symbolic Computation*, 20:169–177, 1995.

[39] J. van der Hoeven. *Asymptotique automatique*. PhD thesis, École Polytechnique, Laboratoire d'Informatique, École Polytechnique, Paris, France, 1997.

[40] J. van der Hoeven. Zero–testing, witness conjectures and differential diophantine approximation. Technical Report 2001-62, Université de Paris–Sud, Orsay, 2001.

[41] D. Zwillinger. *Handbook of Differential Equations*. Academic Press, 1989.