Combining Effects and Coeffects via Grading

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What are coeffects?

*Computations make changes **to** the context (effects)*
*and make demands **from** the context (coeffects).*

- Per-variable coeffect dependency properties:
  - Live-variable analysis (live/dead)
  - Information flow, e.g., security levels (High/Low)
  - Bounded Linear Logic (how much variable reuse? resource analysis)
  - Deconstructors (strictness analysis)
  - Probabilities and schedules
- Whole-context coeffect dependencies (**not in this paper**)
  - Additional hardware resources
  - Library versions
  - Type class instances

Coeffect analysis and meta languages

- Type-and-coeffect system
  - ➡ whole-context analysis [Petricek et al. '13]

    $\Gamma \mid r \vdash e : \tau$
  - ➡ per-variable analysis [Petricek et al. '14, Ghica & Smith '14]

    $x : A \cdot r, \ \ldots, \ y : B \cdot s \vdash e : \tau$

- Coeffect meta-language [Brunel et al. '14]
  - ➡ indexed type constructor, generalises linear exponential

    $\Gamma, x : !_r A \vdash \ e : !_s A$

- Semiring structure on coeffect information
- Semantics via **graded comonads**

This paper: effect and coeffect interactions

- Consider two examples:
  - Security levels masking non-determinism
  - Exceptions change linear reuse bounds

- Core calculus
  - linear-lambda calculus with effect and coeffect types

    $$\Gamma \vdash t : T_e A \qquad\qquad \Gamma, x : \Box_r A \vdash t : B$$

  - parametric in effect, coeffect algebra

- **Graded** semantics: graded monads, graded comonads, graded distributive laws (new here)

Graded semantics

Core calculus

Interaction example

Graded semantics

Core calculus

Interaction example

$$\text{comp}\ \frac{A \to TB \qquad B \to TC}{A \to TC} \qquad \text{id}\ \frac{}{A \to TA}$$

monads

comp $\dfrac{A \to T_{e} B \qquad \cdots}{A \to T \cdots}$

with monoid $(E, \bullet, 1)$ ...f]

Graded slogan:
match the structure of computation with structure in the indices

comp $\dfrac{\Box_{r} A \to B \qquad \Box_{s} B \to C}{\Box_{r * s} A \to C}$ id $\dfrac{}{\Box_{1} A \to A}$

with monoid $(R, *, 1)$ graded comonad [coeffect papers]

Graded semantics

Core calculus

Interaction example

Graded semantics

Core calculus

Interaction example

Core calculus

i. Graded monadic meta language for effects

ii. Graded comonadic meta language for coeffects

iii. Linear lambda calculus (substructural via coeffects)

iv. Graded distributed law for interaction

1. Graded monadic meta language for effects

Pre-ordered monoid of effect information $E = (\mathcal{E}, \bullet, 1, \leq)$

$$\text{letT}\ \frac{\Gamma \vdash t_1 : T_e A \qquad \Delta, x : A \vdash t_2 : T_f B}{\Gamma + \Delta \vdash \textbf{let}\ \langle x \rangle = t_1\ \textbf{in}\ t_2 : T_{e \bullet f} B} \qquad \text{unit}\ \frac{\Gamma \vdash t : A}{\Gamma \vdash \langle t \rangle : T_1 A}$$

e.g., exceptions, with simple analysis $\mathcal{E} = \{\top, \bot\}$

$1 = \top$

$\bullet = \wedge$

$\bot$ = definite exception

$\top$ = may succeed

$\bot \wedge \bot = \bot$

$\bot \wedge \top = \bot$

$\top \wedge \bot = \bot$

$\top \wedge \top = \top$

2. Graded comonad meta language for coeffects

Pre-ordered semiring of coeffect information $C = (\mathcal{C}, *, 1, +, 0, \leq)$

$$\text{der}\,\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : \Box_1 A \vdash t : B} \qquad \text{pr}\,\frac{\Box\Gamma \vdash t : B}{r * \Box\Gamma \vdash [t] : \Box_r B}$$

$$\text{let}\Box\,\frac{\Gamma \vdash t_1 : \Box_r A \qquad \Delta, x : \Box_r A \vdash t_2 : B}{\Gamma + \Delta \vdash \textbf{let}\ [x] = t_1\ \textbf{in}\ t_2 : B}$$

e.g. Bounded Linear Logic (Girard et al.) $C = (\mathbb{N}, *, 1, +, 0, \leq)$

$$\text{pr}\,\frac{x : [\mathsf{int}]_1, y : [\mathsf{int}]_2 \vdash x + y + y : \mathsf{int}}{x : [\mathsf{int}]_2, y : [\mathsf{int}]_4 \vdash [x + y + y] : \Box_2 \mathsf{int}}$$

3. Linear lambda calculus

$$\text{ax}\,\frac{}{x : A \vdash x : A} \qquad \text{abs}\,\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \text{app}\,\frac{\Gamma \vdash t : A \to B \quad \Delta \vdash t' : A}{\Gamma + \Delta \vdash t\, t' : B}$$

Coeffect $+$ for contraction $(\mathrm{let}\Box)$, $(\mathrm{let}\mathrm{T})$, (app)

$$(x : A, \Gamma) + (x : A, \Delta) \quad \textit{is ill-formed}$$

$$(x : \Box_r A, \Gamma) + (x : \Box_s A, \Delta) \quad = \quad x : \Box_{r+s} A, (\Gamma + \Delta)$$

Coeffect 0 for weakening (sub)

$$\text{sub}\,\frac{\Gamma \vdash t : A \qquad \Gamma' <: \Gamma \qquad A <: B}{\Gamma', \Box_0 \Delta \vdash t : B}$$

Graded semantics

Core calculus

Interaction example

Graded semantics

Core calculus

Interaction example

BLL / exception composition

(definite exception)

$$\text{pr}\dfrac{\text{der*}\dfrac{\Gamma \vdash t_1 : T_{\perp} A}{\Box_1 \Gamma \vdash t_1 : T_{\perp} A}}{\Box_r \Gamma \vdash [t_1] : \Box_r T_{\perp} A}$$

$$\dfrac{\text{unit}\dfrac{\Delta, x : \Box_r A \vdash t_2 : B}{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}}{\Delta, x' : T_{\perp} \Box_r A \vdash \quad \ldots \quad : T_{\perp} B}$$

Goal: an exception "cancels out" resource requirements
i.e. (above) $r = 1$

BLL / exception composition

(definite exception)

$$\text{der*}\ \frac{\Gamma \vdash t_1 : T_\bot A}{\Box_1 \Gamma \vdash t_1 : T_\bot A} \qquad \text{unit}\ \frac{\Delta, x : \Box_r A \vdash t_2 : B}{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}$$

$$\text{pr}\ \frac{\Box_1 \Gamma \vdash t_1 : T_\bot A}{\Box_1 \Gamma \vdash [t_1] : \Box_1 T_\bot A} \qquad \frac{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}{\Delta, x' : T_\bot \Box_r A \vdash \ \dots\ : T_\bot B}$$

$\neq$

Goal: an exception "cancels out" resource requirements

🔔 `sequence :: Monad m => [m a] → m [a]`

Notion of a *distributive law* of comonad over monad provides:

$$\textbf{dist} : \Box\, T\, A \to T\, \Box\, A$$

Indexed distributive law? **No effect and coeffect interaction**

$$\textbf{dist}_{r,e,A} : \Box_r\, T_e\, A \to T_e\, \Box_r\, A \quad ✗$$

Indexed vs. graded distributive law

Indexed: $\mathbf{dist}_{r,e,A} : \Box_r\, T_e\, A \to T_e\, \Box_r\, A$

Graded slogan:
algebraic structure on indices matching shape of computation

distributive law on monoids: *"matched pair" (Zappa-Szep product)*

$$\iota : R \times E \to R \qquad \langle \kappa, \iota \rangle : R \times E \to E \times R$$

$$\kappa : R \times E \to E$$

$$\iota(r, 1) = r \qquad \iota(r, e \bullet f) = \iota(\iota(r, e), f)$$

$$\iota(1, e) = 1 \qquad \iota(r * s, e) = \iota(r, \kappa(s, e)) * \iota(s, e)$$

$$\kappa(1, e) = e \qquad \kappa(r * s, e) = \kappa(r, \kappa(s, e))$$

$$\kappa(r, 1) = 1 \qquad \kappa(r, e \bullet f) = \kappa(r, e) \bullet \kappa(\iota(r, e), f)$$

Graded: $\mathbf{dist}_{r,e,A} : \Box_{\iota(r,e)}\, T_e\, A \to T_{\kappa(r,e)}\, \Box_r\, A$

Primitive in our core calculus

BLL / exception composition

$$\text{pr}\dfrac{\text{der*}\dfrac{\Gamma \vdash t_1 : T_{\bot} A}{\Box_1 \Gamma \vdash t_1 : T_{\bot} A}}{\Box_1 \Gamma \vdash [t_1] : \Box_1 T_{\bot} A}$$

$$\dfrac{\text{unit}\dfrac{\Delta, x : \Box_r A \vdash t_2 : B}{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}}{\Delta, x' : T_{\bot} \Box_r A \vdash \ \ldots \ : T_{\bot} B}$$

Goal: an exception "cancels out" resource requirements

- Matched pair

$$\kappa(r, e) = e$$

$$\mathbf{dist}_{r,e,A} : \Box_{\iota(r,e)} T_e A \to T_{\kappa(r,e)} \Box_r A$$

BLL / exception composition

$$\text{pr}\,\dfrac{\text{der*}\,\dfrac{\Gamma \vdash t_1 : T_{\bot} A}{\Box_1 \Gamma \vdash t_1 : T_{\bot} A}}{\Box_1 \Gamma \vdash [t_1] : \Box_1 T_{\bot} A}$$

$$\dfrac{\text{unit}\,\dfrac{\Delta, x : \Box_r A \vdash t_2 : B}{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}}{\Delta, x' : T_{\bot} \Box_r A \vdash \ \ldots \ : T_{\bot} B}$$

- Matched pair

$$\kappa(r, e) = e$$
$$\iota(r, \bot) = 1$$

$$\mathbf{dist}_{r,e,A} : \Box_{\iota(r,e)} T_e A \to T_e \Box_r A$$

Goal: an exception "cancels out" resource requirements

BLL / exception composition

$$
\cfrac{\cfrac{\cfrac{\Gamma \vdash t_1 : T_{\bot} A}{\Box_1 \Gamma \vdash t_1 : T_{\bot} A}\,\text{der*}}{\Box_1 \Gamma \vdash [t_1] : \Box_1 T_{\bot} A}\,\text{pr}}{\Box_1 \Gamma \vdash \mathbf{dist}\ [t_1] : T_{\bot} \Box_r A}
\qquad
\cfrac{\cfrac{\Delta, x : \Box_r A \vdash t_2 : B}{\Delta, x : \Box_r A \vdash \langle t_2 \rangle : T_1 B}\,\text{unit}}{\Delta, x' : T_{\bot} \Box_r A \vdash \ \ldots\ : T_{\bot} B}
$$

$$
\overline{\Delta + \Box_1 \Gamma \vdash \mathbf{let}\ x' = \mathbf{dist}\ [t_1]\ \mathbf{in} \ldots : T_{\bot} A}
$$

- Matched pair

$$\kappa(r, e) = e$$
$$\iota(r, \bot) = 1$$
$$\iota(r, \top) = r$$

$$\mathbf{dist}_{r,\bot,A} : \Box_1\, T_{\bot}\, A \to T_{\bot}\, \Box_r\, A$$
$$\mathbf{dist}_{r,\top,A} : \Box_r\, T_{\top}\, A \to T_{\top}\, \Box_r\, A$$

Goal: an exception "cancels out" resource requirements ✔

4. Graded distributive law *design space*

"T□" vs "□T"

$$\Box\, T\, A \to T\, \Box\, A \qquad\qquad T\, \Box\, A \to \Box\, T\, A$$

"forward" (L) vs "backward" (R)

$$T_{e}\, \Box_{\dots}\, A \to \Box_{\dots}\, T_{\kappa(e,r)}\, A \qquad\qquad T_{\dots}\, \Box_{r}\, A \to \Box_{\iota(r,e)}\, T_{\dots}\, A$$

$$T_{\kappa(e,r)}\, \Box_{\dots}\, A \to \Box_{\dots}\, T_{e}\, A \qquad\qquad T_{\dots}\, \Box_{\iota(r,e)}\, A \to \Box_{r}\, T_{\dots}\, A$$

ϕ	T over $\Box$ ($T\Box$)					$\Box$ over T ($\Box T$)				
	$F^{\phi}_{r,e}$			$G^{\phi}_{r,e}$		$F^{\phi}_{r,e}$			$G^{\phi}_{r,e}$	
LL	T_{e}	$\Box_{r}$	$\to$	$\Box_{\iota(r,e)}$	$T_{\kappa(r,e)}$	$\Box_{r}$	T_{e}	$\to$	$T_{\kappa(r,e)}$	$\Box_{\iota(r,e)}$
RR	$T_{\kappa(r,e)}$	$\Box_{\iota(r,e)}$	$\to$	$\Box_{r}$	T_{e}	$\Box_{\iota(r,e)}$	$T_{\kappa(r,e)}$	$\to$	T_{e}	$\Box_{r}$
LR	T_{e}	$\Box_{\iota(r,e)}$	$\to$	$\Box_{r}$	$T_{\kappa(r,e)}$	$\Box_{\iota(r,e)}$	T_{e}	$\to$	$T_{\kappa(r,e)}$	$\Box_{r}$
RL	$T_{\kappa(r,e)}$	$\Box_{r}$	$\to$	$\Box_{\iota(r,e)}$	T_{e}	$\Box_{r}$	$T_{\kappa(r,e)}$	$\to$	T_{e}	$\Box_{\iota(r,e)}$

$$\vdash \mathbf{dist}^{\phi} : F^{\phi}_{r,e} A \to G^{\phi}_{r,e} A$$

Categorical semantics

- **Graded strong monad** (Katsumata 2014)
- **Graded monoidal comonad** (Petricek et al. 2013, Brunel et al. 2014, Petricek et al. 2014, Ghica&Smith et al. 2014)
- **Family of matched-pair graded distributive law** (new)
- **Soundness theorem**

Concluding remarks

- Foundation for considering effect-coeffect interaction
- Further possibilities?
  - Exceptions interacting with security levels (e.g., content of a high security exception not observable to lower security observer)
  - Non-determinism with probabilistic variables
- Graded distributive laws useful in other contexts

$$\texttt{sequence} :: \quad [\texttt{m a}] \to \texttt{m [a]}$$
$$\texttt{sequence} :: \quad [\texttt{m}_e \ \texttt{a}]_n \to \texttt{m}_{e^n} [\texttt{a}]_n$$

Thank you.

Extras

Graded distributive law

$$\frac{D_r A \xrightarrow{f} T_e B \qquad D_s B \xrightarrow{g} T_f C}{D_{r*\iota(s,e)} A \xrightarrow{f^\dagger} D_{\iota(s,e)} T_e B \xrightarrow{\sigma_{s,e,A}} T_{\kappa(s,e)} D_s B \xrightarrow{g^*} T_{\kappa(s,e)\bullet f} C}$$

backward propagation of coeffects forward propagation of effects

distributive law on monoids: *"matched pair" (Zappa-Szep product)*

(monoid on $E \times R$)

$$\iota : R \times E \to R$$
$$\kappa : R \times E \to E$$

$$\langle \kappa, \iota \rangle : R \times E \to E \times R$$

$$\iota(r, 1) = r \qquad \iota(r, e \bullet f) = \iota(\iota(r, e), f)$$
$$\iota(1, e) = 1 \qquad \iota(r * s, e) = \iota(r, \kappa(s, e)) * \iota(s, e)$$
$$\kappa(1, e) = e \qquad \kappa(r * s, e) = \kappa(r, \kappa(s, e))$$
$$\kappa(r, 1) = 1 \qquad \kappa(r, e \bullet f) = \kappa(r, e) \bullet \kappa(\iota(r, e), f)$$